4 Geometric Interpretations

4.1 Introduction

In Chapter 2, the exterior product operation was introduced onto the elements of a linear space $\underset{1}{\Lambda}$, enabling the construction of a series of new linear spaces $\underset{m}{\Lambda}$ possessing new types of elements. In this chapter, the elements of $\underset{1}{\Lambda}$ will be *interpreted*, some as *vectors*, some as *points*. This will be done by singling out one particular element of $\underset{1}{\Lambda}$ and conferring upon it the *interpretation* of *origin point*. All the other elements of the linear space then divide into two categories. Those that involve the origin point will be called (weighted) points, and those that do not will be called vectors. As this distinction is developed in succeeding sections it will be seen to be both illuminating and consistent with accepted notions. Vectors and points will be called *geometric interpretations* of the elements of $\underset{1}{\Lambda}$.

Some of the more important consequences however, of the distinction between vectors and points arise from the distinctions thereby generated in the higher grade spaces $\underset{m}{\Lambda}$. It will be shown that a simple element of $\underset{m}{\Lambda}$ takes on two interpretations. The first, that of a *multivector* (or *m-vector*) is when the *m*-element can be expressed in terms of vectors alone. The second, that of a *bound multivector* is when the *m*-element requires both points and vectors to express it. These simple interpreted elements will be found useful for defining geometric entities such as lines and planes and their higher dimensional analogues known as *multiplanes* (or *m-planes*). Unions and intersections of multiplanes may then be calculated straightforwardly by using the bound multivectors which define them. A multivector may be visualized as a 'free' entity with no location. A bound multivector may be visualized as 'bound' through a location in space.

It is not only simple interpreted elements which will be found useful in applications however. In Chapter 8: Exploring Screw Algebra and Chapter 9: Exploring Mechanics, a basis for a theory of mechanics is developed whose principal quantities (for example, systems of forces, momentum of a system of particles, velocity of a rigid body etc.) may be represented by a general interpreted 2-element, that is, by the *sum* of a bound vector and a bivector.

In the literature of the nineteenth century, wherever vectors and points were considered together, vectors were introduced as point differences. When it is required to designate physical quantities it is not satisfactory that all vectors should arise as the differences of points. In later literature, this problem appeared to be overcome by designating points by their position vectors alone, making vectors the fundamental entities [Gibbs 1886]. This approach is not satisfactory either, since by excluding points much of the power of the calculus for dealing with free and located entities *together* is excluded. In this book we do not require that vectors be defined in terms of points, but rather, postulate a difference of interpretation between the origin element and those elements not involving the origin. This approach permits the existence of points and vectors together without the vectors necessarily arising as point differences.

In this chapter, as in the preceding chapters, $\underset{1}{\Lambda}$ and the spaces $\underset{m}{\Lambda}$ do not yet have a metric. That is, there is no way of calculating a measure or magnitude associated with an element. The interpretation discussed therefore may also be supposed non-metric. In the next chapter a metric will be introduced onto the uninterpreted spaces and the consequences of this for the interpreted elements developed.

In summary then, it is the aim of this chapter to set out the distinct non-metric geometric interpretations of *m*-elements brought about by the interpretation of one specific element of $\underset{1}{\Lambda}$ as an origin point.

4.2 Geometrically Interpreted 1-elements

Vectors

The most common current geometric interpretation for an element of a linear space is that of a *vector*. We suppose in this chapter that the linear space does not have a metric (that is, we cannot calculate magnitudes). Such a vector has the geometric properties of *direction* and *sense* (but no *location*), and will be graphically *depicted* by a directed and sensed line segment thus:

Graphic showing an arrow.

This depiction is unsatisfactory in that the line segment has a definite location and length whilst the vector it is depicting is supposed to possess neither of these properties. Nevertheless we are not aware of any more faithful alternative.

A linear space, all of whose elements are interpreted as vectors, will be called a *vector space*. The geometric interpretation of the addition operation of a vector space is the parallelogram rule.

Since an element of $\underset{1}{\Lambda}$ may be written $\mathbf{a}\wedge\mathbf{x}$ where $\mathbf{a}$ is a scalar, then the interpretation of $\mathbf{a}\wedge\mathbf{x}$ as a vector suggests that it may be viewed as the product of two quantities.

1. A direction factor $\mathbf{x}$.
2. An intensity factor $\mathbf{a}$.

The term 'intensity' was coined by Alfed North Whitehead [ ]. The sign of $\mathbf{a}$ may be associated with the sense of the vector. Of course this division of vector into direction and intensity factors is arbitrarily definable, but is conceptually useful in non-metric spaces. In a non-metric space, two vectors may be compared if and only if they have the same direction. Then it is their scalar intensities that are compared. Intensity should not be confused with the metric quantity of length or magnitude.

Points

In order to describe *position* in space it is necessary to have a *reference point*. This point is usually called the *origin*.

Rather than the standard technique of implicitly assuming the origin and working only with vectors to describe position, we find it important for later applications to *augment the vector space with the origin as a new element* to create a new linear space with one more element in its basis. For reasons which will appear later, such a linear space will be called a *bound vector space*.

The only difference between the origin element and the vector elements of the linear space is their interpretation. The origin element is interpreted as a point. We will denote it in *GrassmannAlgebra* by $\mathbb{O}$, which we can access from the palette or type as ⎋dsO⎋ (double-struck capital O). The bound vector space in addition to its vectors and its origin now possesses a new set of elements requiring interpretation: those formed from the sum of the origin and a vector.

$$P == \mathbb{O} + x$$

It is these elements that will be used to describe position and that we will call *points*. The vector x is called the *position vector* of the point P.

It follows immediately from this definition of a point that *the difference of two points is a vector*:

$$P_1 - P_2 == (\mathbb{O} + x_1) - (\mathbb{O} + x_2) == x_1 - x_2$$

Remember that the bound vector space does not yet have a metric. That is, the distance between two points (the measure of the vector equal to the point difference) is not meaningful.

The *sum of a point and a vector* is another point.

$$P + y == \mathbb{O} + x + y == \mathbb{O} + (x + y)$$

and so a vector may be viewed as a *carrier of points*. That is, the addition of a vector to a point carries or transforms it to another point.

A scalar multiple of a point (of the form $a\,p$ or $a \wedge p$) will be called a *weighted point*. Weighted points are summed just like a set of point masses is deemed equivalent to their total mass situated at their centre of mass. For example:

$$\sum m_i\, P_i == \sum m_i\, (\mathbb{O} + x_i) == \left(\sum m_i\right) \left(\mathbb{O} + \frac{\Sigma\, m_i\, x_i}{\Sigma\, m_i}\right)$$

This equation may be interpreted as saying that the sum of a number of mass-weighted points $\sum m_i\, p_i$ is equivalent to the centre of gravity $\mathbb{O} + \frac{\Sigma\, m_i\, x_i}{\Sigma\, m_i}$ weighted by the total mass $\sum m_i$.

As will be seen in Section 4.4 below, a weighted point may also be viewed as a bound scalar.

Historical Note

Sir William Rowan Hamilton in his *Lectures on Quaternions* [Hamilton 1853] was the first to introduce the notion of vector as 'carrier' of points.

> ... I regard the symbol **B-A** as denoting "the *step* from **B** to **A**": namely, that step by making which, from the given point **A**, we should reach or arrive at the sought point **B**; and so determine, generate, mark or *construct* that point. This step, (which we always suppose to be a straight line) may also in my opinion be properly called a *vector*; or more fully, it may be called "the vector of the point **B** from the point **A**": because it may be considered as having for its office, function, work, task or business, to transport or *carry* (in Latin *vehere*) a moveable point, from the given or initial position **A**, to the sought or final position **B**.

A shorthand for declaring standard bases

Any geometry that we do with points will require us to declare the origin $\mathbb{O}$ as one of the elements of the basis. We have already seen that a shorthand way of declaring a basis $\{e_1, e_2, \cdots, e_n\}$ is by entering $\mathbb{V}_n$. Declaring the augmented basis $\{\mathbb{O}, e_1, e_2, \cdots, e_n\}$ can be accomplished by entering $\mathbb{P}_n$. These are double-struck capital letters subscripted with the integer *n* denoting the desired 'vectorial' dimension of the space. For example, entering $\mathbb{V}_3$ or $\mathbb{P}_3$:

$\mathbb{V}_3$

$\{e_1, e_2, e_3\}$

$\mathbb{P}_3$

$\{\mathbb{O}, e_1, e_2, e_3\}$

We may often precede a calculation with one of these followed by a semi-colon. This accomplishes the declaration of the basis but for brevity suppresses the confirming output. For example:

$\mathbb{P}_2$; **Basis∧[]**

$\{1, \mathbb{O}, e_1, e_2, \mathbb{O}\wedge e_1, \mathbb{O}\wedge e_2, e_1\wedge e_2, \mathbb{O}\wedge e_1\wedge e_2\}$

Example: Calculation of the centre of mass

Suppose a space $\mathbb{P}_3$ with basis $\{\mathbb{O}, e_1, e_2, e_3\}$ and a set of masses M_i situated at points P_i. It is required to find their centre of mass. First declare the basis, then enter the mass points.

$\mathbb{P}_3$

$\{\mathbb{O}, e_1, e_2, e_3\}$

$$M_1 = 2\,P_1;\quad P_1 = \mathbb{O} + e_1 + 3\,e_2 - 4\,e_3;$$
$$M_2 = 4\,P_2;\quad P_2 = \mathbb{O} + 2\,e_1 - e_2 - 2\,e_3;$$
$$M_3 = 7\,P_3;\quad P_3 = \mathbb{O} - 5\,e_1 + 3\,e_2 - 6\,e_3;$$
$$M_4 = 5\,P_4;\quad P_4 = \mathbb{O} + 4\,e_1 + 2\,e_2 - 9\,e_3;$$

We simply add the mass-weighted points.

$$M = \sum_{i=1}^{4} M_i$$

$$5\,(\mathbb{O} + 4\,e_1 + 2\,e_2 - 9\,e_3) + 7\,(\mathbb{O} - 5\,e_1 + 3\,e_2 - 6\,e_3) + 2\,(\mathbb{O} + e_1 + 3\,e_2 - 4\,e_3) + 4\,(\mathbb{O} + 2\,e_1 - e_2 - 2\,e_3)$$

Simplifying this gives a weighted point with weight 18, the scalar attached to the origin.

```
M = Simplify[M]
```

$$18\,\mathbb{O} - 5\,e_1 + 33\,e_2 - 103\,e_3$$

Taking the weight out as a factor, that is, expressing the result in the form *mass* × *point*

```
M = WeightedPointForm[M]
```

$$18\left(\mathbb{O} + \frac{1}{18}\,(-5\,e_1 + 33\,e_2 - 103\,e_3)\right)$$

Thus the total mass is 18 situated at the point $\mathbb{O} + \frac{1}{18}\,(-5\,e_1 + 33\,e_2 - 103\,e_3)$.

4.3 Geometrically Interpreted 2-Elements

Simple geometrically interpreted 2-elements

It has been seen in Chapter 2 that the linear space $\underset{2}{\Lambda}$ may be generated from $\underset{1}{\Lambda}$ by the exterior product operation. In the preceding section the elements of $\underset{1}{\Lambda}$ have been given two geometric interpretations: that of a *vector* and that of a *point*. These interpretations in turn generate various other interpretations for the elements of $\underset{2}{\Lambda}$.

In $\underset{2}{\Lambda}$ there are at first sight three possibilities for simple elements:

1. $x\wedge y$ (vector by vector).
2. $P\wedge x$ (point by vector).
3. $P_1 \wedge P_2$ (point by point).

However, $P_1 \wedge P_2$ may be expressed as $P_1 \wedge (P_2 - P_1)$ which is a point by a vector and thus reduces to the second case.

There are thus two *simple* interpreted elements in $\underset{2}{\Lambda}$:

1. $x\wedge y$ (*the simple bivector*).
2. $P\wedge x$ (*the bound vector*).

A note on terminology

The term *bound* as in the *bound vector* **P∧x** indicates that the vector **x** is conceived of as bound *through* the point **P**, rather than *to* the point **P**, since the latter conception would give the incorrect impression that the vector was located *at* the point **P**. By adhering to the terminology *bound through*, we get a slightly more correct impression of the 'freedom' that the vector enjoys.

The term 'bound vector' is often used in engineering to specify the type of quantity a force is.

The bivector

Earlier in this chapter, a vector **x** was depicted graphically by a directed and sensed line segment supposed to be located nowhere in particular.

Graphic showing an arrow with symbol **x** attached.

In like manner a simple bivector may be depicted graphically by an *oriented plane segment* also supposed located nowhere in particular.

Graphic showing a parallelogram constructed from two vectors **x** and **y** with common tails. The symbol **x∧y** is attached to the parallelogram.

Orientation is a relative concept. The plane segment depicting the bivector **y∧x** is of opposite orientation to that depicting **x∧y**.

Graphic showing a parallelogram **y∧x** .

The oriented plane segment or parallelogram depiction of the simple bivector is misleading in two main respects. It incorrectly suggests a specific *location* in the plane and *shape* of the parallelogram. Indeed, since **x∧y** == **x∧(x+y)**, another valid depiction of this simple bivector would be a parallelogram with sides constructed from vectors **x** and **x+y**.

Graphic showing parallelograms **x∧y** and **x∧(x+y)** superimposed.

In the following chapter a metric will be introduced onto $\underset{1}{\Lambda}$ from which a metric is induced onto $\underset{2}{\Lambda}$. This will permit the definition of the measure of a vector (its length) and the measure of the simple bivector (its area).

The measure of a simple bivector is geometrically interpreted as the *area of the parallelogram* formed by any two vectors in terms of which the simple bivector can be expressed. For example, the area of the parallelograms in the previous two figures are the same. From this point of view the parallelogram depiction is correctly *suggestive*, although the parallelogram is not of fixed shape. However, a bivector is as independent of the vector factors used to express it as any area is of its shape. Strictly speaking therefore, a bivector may be interpreted as a portion of an (unlocated) plane of any shape. In a metric space, this portion will have an area.

Earlier in the chapter it was remarked that a vector may be viewed as a 'carrier' of points. Analogously, a simple bivector may be viewed as a *carrier of bound vectors*. This view will be more fully explored in the next section.

A *sum of simple bivectors* is called a *bivector*. In two and three dimensions all bivectors are simple. This will have important consequences for our exploration of screw algebra in Chapter 7 and its application to mechanics in Chapter 8.

The bound vector

In mechanics the concept of force is paramount. In Chapter 9: Exploring Mechanics we will show that a force may be represented by a bound vector, and that a system of forces may be represented by a sum of bound vectors.

It has already been shown that a bound vector may be expressed either as the product of a point with a vector or as the product of two points.

$$P_1 \wedge x == P_1 \wedge (P_2 - P_1) == P_1 \wedge P_2 \qquad P_2 == P_1 + x$$

The bound vector in the form $P_1 \wedge x$ defines a line through the point P_1 in the direction of x. Similarly, in the form $P_1 \wedge P_2$ it defines the line through P_1 and P_2.

Graphic showing
1. A line through a point parallel to a vector.
2. A line through two points.
Both lines are parallel.

It is an important property of the Grassmann algebra that *any* point on this line may be used to represent the bound vector. Since if P and P^* are any two points in the line, $P-P^*$ is a vector of the same direction as x so that:

$$(P - P^*) \wedge x == 0 \quad \Longrightarrow \quad P \wedge x == P^* \wedge x$$

A bound vector may thus be depicted by a line, a point on it, and a vector parallel to it. For compactness, we will usually depict the vector of the bound vector as lying in the line.

Graphic showing a line through a point with an arrow in the line (not attached to the point).

This graphical depiction is misleading in that it suggests that the vector x has a specific location, and that the point P is of specific importance over other points in the line. Nevertheless we are not aware of any more faithful alternative.

It has been mentioned in the previous section that a simple bivector may be viewed as a 'carrier' of bound vectors. To see this, take any bound vector $P \wedge x$ and a bivector whose space contains x. The bivector may be expressed in terms of x and some other vector, y say, yielding $y \wedge x$. Thus:

$$P \wedge x + y \wedge x == (P + y) \wedge x == P^* \wedge x$$

Graphic showing a simple bivector added to a bound vector to give another bivector.

The geometric interpretation of the addition of such a simple bivector to a bound vector is then similar to that for the addition of a vector to a point, that is, a shift in position.

Sums of bound vectors

A *sum of bound vectors* $\Sigma\, P_i \wedge x_i$ (except in the case $\Sigma\, x_i == 0$) may always be reduced to the *sum of a bound vector and a bivector*, since, by choosing an *arbitrary* point P, $\Sigma\, P_i \wedge x_i$ may always be written in the form:

$$\Sigma\, P_i \wedge x_i == P \wedge (\Sigma\, x_i) + \Sigma\, (P_i - P) \wedge x_i \qquad 4.1$$

If $\Sigma\, x_i == 0$ then the sum is a bivector. This transformation is of fundamental importance in our exploration of mechanics in Chapter 8.

Example: Reducing a sum of bound vectors

In this example we verify that a sum of any number of bound vectors can always be reduced to the sum of a bound vector and a bivector. Note however that it is only in two or three vector dimensions that the bivector is necessarily simple. We begin by declaring the bound vector space of three vector dimensions, $\mathbb{P}_3$.

$$\mathbb{P}_3$$

$$\{\mathbb{O}, e_1, e_2, e_3\}$$

Next, we define, enter, then sum four bound vectors.

$$\beta_1 = P_1 \wedge x_1;\quad P_1 = \mathbb{O} + e_1 + 3\, e_2 - 4\, e_3;\quad x_1 = e_1 - e_3;$$
$$\beta_2 = P_2 \wedge x_2;\quad P_2 = \mathbb{O} + 2\, e_1 - e_2 - 2\, e_3;\quad x_2 = e_1 - e_2 + e_3;$$
$$\beta_3 = P_3 \wedge x_3;\quad P_3 = \mathbb{O} - 5\, e_1 + 3\, e_2 - 6\, e_3;\quad x_3 = 2\, e_1 + 3\, e_2;$$
$$\beta_4 = P_4 \wedge x_4;\quad P_4 = \mathbb{O} + 4\, e_1 + 2\, e_2 - 9\, e_3;\quad x_4 = 5\, e_3;$$

$$B = \sum_{i=1}^{4} \beta_i$$

$$(\mathbb{O} + 4\, e_1 + 2\, e_2 - 9\, e_3) \wedge (5\, e_3) + (\mathbb{O} - 5\, e_1 + 3\, e_2 - 6\, e_3) \wedge (2\, e_1 + 3\, e_2) + (\mathbb{O} + e_1 + 3\, e_2 - 4\, e_3) \wedge (e_1 - e_3) + (\mathbb{O} + 2\, e_1 - e_2 - 2\, e_3) \wedge (e_1 - e_2 + e_3)$$

By expanding these products, simplifying and collecting terms, we obtain the sum of a bound vector (through the origin) $\mathbb{O} \wedge (4\, e_1 + 2\, e_2 + 5\, e_3)$ and a bivector $-25\, e_1 \wedge e_2 + 39\, e_1 \wedge e_3 + 22\, e_2 \wedge e_3$. We can use `GrassmannSimplify` to do the computations for us.

$$\mathcal{G}[B]$$

$$\mathbb{O} \wedge (4\, e_1 + 2\, e_2 + 5\, e_3) - 25\, e_1 \wedge e_2 + 39\, e_1 \wedge e_3 + 22\, e_2 \wedge e_3$$

We could just as well have expressed this 2-element as bound through (for example) the point $\mathbb{O} + e_1$. To do this, we simply add $e_1 \wedge (4\, e_1 + 2\, e_2 + 5\, e_3)$ to the bound vector and subtract it from the bivector to get:

$$(\mathbb{O} + e_1) \wedge (4\, e_1 + 2\, e_2 + 5\, e_3) - 27\, e_1 \wedge e_2 + 34\, e_1 \wedge e_3 + 22\, e_2 \wedge e_3$$

4.4 Geometrically Interpreted *m*-elements

Types of geometrically interpreted *m*-elements

In $\underset{m}{\Lambda}$ the situation is analogous to that in $\underset{2}{\Lambda}$. A simple product of *m* points and vectors may always be reduced to the product of a point and a simple (*m*–1)-vector by subtracting one of the points from all of the others. For example, take the exterior product of three points and two vectors. By subtracting the first point, say, from the other two, we can cast the product into the form of a bound simple 4-vector.

$$P \wedge P_1 \wedge P_2 \wedge x \wedge y == P \wedge (P_1 - P) \wedge (P_2 - P) \wedge x \wedge y$$

There are thus only two *simple* interpreted elements in $\underset{m}{\Lambda}$:

1. $x_1 \wedge x_2 \wedge \cdots x_m$ (the simple *m*-vector).
2. $P \wedge x_2 \wedge \cdots x_m$ (the bound simple (*m*–1)-vector).

A sum of simple *m*-vectors is called an *m*-vector and the non-simple interpreted elements of $\underset{m}{\Lambda}$ are sums of *m*-vectors.

If $\underset{m}{\alpha}$ is a (not necessarily simple) *m*-vector, then $P \wedge \underset{m}{\alpha}$ is called a *bound m-vector*.

A sum of bound *m*-vectors may always be reduced to the sum of a bound *m*-vector and an (*m*+1)-vector.

These interpreted elements and their relationships will be discussed further in the following sections.

The *m*-vector

The simple *m*-vector, or *multivector*, is the multidimensional equivalent of the vector. As with a vector, it does not have the property of location. The *m*-dimensional vector space of a simple *m*-vector may be used to define the multidimensional *direction* of the *m*-vector.

A simple *m*-vector may be depicted by an *oriented m-space segment*. A 3-vector may be depicted by a parallelepiped.

Graphic showing a parallelepiped formed from three vectors.

The orientation is given by the order of the factors in the simple *m*-vector. An interchange of any two factors produces an *m*-vector of opposite orientation. By the anti-symmetry of the exterior product, there are just two distinct orientations.

In Chapter 6: The Interior Product, it will be shown that the measure of a 3-vector may be geometrically interpreted as the volume of this parallelepiped. However, the depiction of the simple 3-vector in the manner above suffers from similar defects to those already described for the bivector: namely, it incorrectly suggests a specific location and shape of the parallelepiped.

A simple *m*-vector may also be viewed as a carrier of bound simple (*m*–1)-vectors in a manner analogous to that already described for the bivector.

A sum of simple *m*-vectors (that is, an *m*-vector) is not necessarily reducible to a simple *m*-vector, except in $\underset{0}{\Lambda}, \underset{1}{\Lambda}, \underset{n-1}{\Lambda}, \underset{n}{\Lambda}$.

The bound *m*-vector

The exterior product of a point and an *m*-vector is called a *bound m-vector*. Note that it belongs to $\underset{m+1}{\Lambda}$. A sum of bound *m*-vectors is not necessarily a bound *m*-vector. However, it may in general be reduced to the sum of a bound *m*-vector and an (*m*+1)-vector as follows:

$$\sum \mathbf{P_i} \wedge \underset{m}{\alpha_i} == \sum (\mathbf{P} + \beta_i) \wedge \underset{m}{\alpha_i} == \mathbf{P} \wedge \sum \underset{m}{\alpha_i} + \sum \beta_i \wedge \underset{m}{\alpha_i} \qquad 4.2$$

The first term $\mathbf{P} \wedge \sum \underset{m}{\alpha_i}$ is a bound *m*-vector providing $\sum \underset{m}{\alpha_i} \neq \mathbf{0}$ and the second term is an (*m*+1)-vector.

When *m* = 0, a bound 0-vector or *bound scalar* $\mathbf{p} \wedge \mathbf{a}$ (= $\mathbf{ap}$) is seen to be equivalent to a weighted point.

When *m* = *n*, any bound *n*-vector is but a scalar multiple of $\mathbb{O} \wedge \mathbf{e_1} \wedge \mathbf{e_2} \wedge \cdots \wedge \mathbf{e_n}$ for any basis.

The graphical depiction of bound simple *m*-vectors presents even greater difficulties than those already discussed for bound vectors. As in the case of the bound vector, the point used to express the bound simple *m*-vector is not unique.

In Section 4.6 we will see how bound simple *m*-vectors may be used to define multiplanes.

Bound simple *m*-vectors expressed by points

A bound simple *m*-vector may always be expressed as a product of *m*+1 points.
Let $\mathbf{P_i} = \mathbf{P_0} + \mathbf{x_i}$, then:

$$\begin{aligned} &\mathbf{P_0} \wedge \mathbf{x_1} \wedge \mathbf{x_2} \wedge \cdots \wedge \mathbf{x_m} \\ &== \mathbf{P_0} \wedge (\mathbf{P_1} - \mathbf{P_0}) \wedge (\mathbf{P_2} - \mathbf{P_0}) \wedge \cdots \wedge (\mathbf{P_m} - \mathbf{P_0}) \\ &== \mathbf{P_0} \wedge \mathbf{P_1} \wedge \mathbf{P_2} \wedge \cdots \wedge \mathbf{P_m} \end{aligned}$$

Conversely, as we have already seen, a product of *m*+1 points may always be expressed as the product of a point and a simple *m*-vector by subtracting one of the points from all of the others.

The *m-vector of a bound simple m-vector* $\mathbf{P_0} \wedge \mathbf{P_1} \wedge \mathbf{P_2} \wedge \cdots \wedge \mathbf{P_m}$ may thus be expressed in terms of these points as:

$$(\mathbf{P_1} - \mathbf{P_0}) \wedge (\mathbf{P_2} - \mathbf{P_0}) \wedge \cdots \wedge (\mathbf{P_m} - \mathbf{P_0})$$

A particularly symmetrical formula results from the expansion of this product reducing it to a form no longer showing preference for P_0.

$$(P_1 - P_0) \wedge (P_2 - P_0) \wedge \cdots \wedge (P_m - P_0) = \sum_{i=0}^{m} (-1)^i P_0 \wedge P_1 \wedge \cdots \wedge \square_i \wedge \cdots \wedge P_m \qquad 4.3$$

Here $\square_i$ denotes deletion of the factor P_i from the product.

4.5 Decomposition into Components

The shadow

In Chapter 3 we developed a formula which expressed a *p*-element as a sum of components, the element being decomposed with respect to a pair of elements $\underset{m}{\alpha}$ and $\underset{n-m}{\beta}$ which together span the whole space. The first and last terms of this decomposition were given as:

$$\underset{p}{x} = \frac{\left(\underset{m}{\alpha} \wedge \underset{p}{x}\right) \vee \underset{n-m}{\beta}}{\underset{m}{\alpha} \vee \underset{n-m}{\beta}} + \cdots + \frac{\underset{m}{\alpha} \vee \left(\underset{p}{x} \wedge \underset{n-m}{\beta}\right)}{\underset{m}{\alpha} \vee \underset{n-m}{\beta}}$$

Grassmann called the last term the *shadow of* $\underset{p}{x}$ *on* $\underset{m}{\alpha}$ *excluding* $\underset{n-m}{\beta}$.

It can be seen that the first term can be rearranged as the *shadow of* $\underset{p}{x}$ *on* $\underset{n-m}{\beta}$ *excluding* $\underset{m}{\alpha}$.

$$\frac{\left(\underset{m}{\alpha} \wedge \underset{p}{x}\right) \vee \underset{n-m}{\beta}}{\underset{m}{\alpha} \vee \underset{n-m}{\beta}} = \frac{\underset{n-m}{\beta} \vee \left(\underset{p}{x} \wedge \underset{m}{\alpha}\right)}{\underset{n-m}{\beta} \vee \underset{m}{\alpha}}$$

If $p = 1$, the decomposition formula reduces to the sum of just two components, x_α and x_β, where x_α lies in $\underset{m}{\alpha}$ and x_β lies in $\underset{n-m}{\beta}$.

$$x = x_\alpha + x_\beta = \frac{\underset{m}{\alpha} \vee \left(x \wedge \underset{n-m}{\beta}\right)}{\underset{m}{\alpha} \vee \underset{n-m}{\beta}} + \frac{\underset{n-m}{\beta} \vee \left(x \wedge \underset{m}{\alpha}\right)}{\underset{n-m}{\beta} \vee \underset{m}{\alpha}} \qquad 4.4$$

We now explore this decomposition with a number of geometric examples, beginning with the simplest case of decomposing a point on a line into a sum of two weighted points in the line.

Decomposition in a 2-space

Decomposition of a vector in a vector 2-space

Suppose we have a vector 2-space defined by the bivector $\alpha \wedge \beta$. We wish to decompose a vector $\mathbf{x}$ in the space to give one component in α and the other in β. Applying the decomposition formula gives:

$$\mathbf{x}_\alpha == \frac{\alpha \vee (\mathbf{x} \wedge \beta)}{\alpha \vee \beta} == \left(\frac{\mathbf{x} \vee \beta}{\alpha \vee \beta}\right) \alpha$$

$$\mathbf{x}_\beta == \frac{\beta \vee (\mathbf{x} \wedge \alpha)}{\beta \vee \alpha} == \left(\frac{\mathbf{x} \vee \alpha}{\beta \vee \alpha}\right) \beta$$

Graphic of two vectors α and β in a plane with a third vector x between them, showing its decomposition in the directions of α and β.

The coefficients of α and β are scalars showing that $\mathbf{x}_\alpha$ is congruent to α and $\mathbf{x}_\beta$ is congruent to β. If each of the three vectors is expressed in basis form, we can determine these scalars more specifically. For example:

$$\mathbf{x} == x_1\, e_1 + x_2\, e_2; \quad \alpha == a_1\, e_1 + a_2\, e_2; \quad \beta == b_1\, e_1 + b_2\, e_2$$

$$\frac{\mathbf{x} \vee \beta}{\alpha \vee \beta} == \frac{(x_1\, e_1 + x_2\, e_2) \vee (b_1\, e_1 + b_2\, e_2)}{(a_1\, e_1 + a_2\, e_2) \vee (b_1\, e_1 + b_2\, e_2)} ==$$
$$\frac{(x_1\, b_2 - x_2\, b_1)\, e_1 \vee e_2}{(a_1\, b_2 - a_2\, b_1)\, e_1 \vee e_2} == \frac{(x_1\, b_2 - x_2\, b_1)}{(a_1\, b_2 - a_2\, b_1)}$$

Finally then we can express the original vector $\mathbf{x}$ as the required sum of two components, one in α and one in β.

$$\mathbf{x} == \frac{(x_1\, b_2 - x_2\, b_1)}{(a_1\, b_2 - a_2\, b_1)} \alpha + \frac{(x_1\, a_2 - x_2\, a_1)}{(b_1\, a_2 - b_2\, a_1)} \beta$$

Decomposition of a point in a line

These same calculations apply to decomposing a point $\mathbf{P}$ into two component weighted points congruent to two given points $\mathbf{Q}$ and $\mathbf{R}$ in a line. Let:

$$\mathbf{P} == \mathbb{O} + p\, e_1; \quad \mathbf{Q} == \mathbb{O} + q\, e_1; \quad \mathbf{R} == \mathbb{O} + r\, e_1;$$

Graphic of two points Q and R in a line with a third point P between them. Q and R are shown with weights attached.

The decomposition may be obtained by following the process *mutatis mutandis*. Alternatively we may substitute directly into the formula above by making the correspondence: $e_1 \to \mathbb{O}$, $e_2 \to e_1$, whence:

$$x_1 \to 1,\ x_2 \to p,\ a_1 \to 1,\ a_2 \to q,\ b_1 \to 1,\ b_2 \to r$$

The resulting formula gives the point **P** as the sum of two weighted points, scalar multiples of the points **Q** and **R**.

$$P == \frac{(r - p)}{(r - q)} Q + \frac{(q - p)}{(q - r)} R$$

Decomposition of a vector in a line

We can also decompose a vector **x** into two component weighted points congruent to two given points **Q** and **R** in a line. Let:

$$x == a\, e_1;\quad Q == \mathbb{O} + q\, e_1;\quad R == \mathbb{O} + r\, e_1;$$

The decomposition formula then shows that the vector **x** may be expressed as the difference of two points, each with the same weight representing the ratio of the vector **x** to the parallel vector **Q−R**.

$$x == \left(\frac{a}{q - r}\right) Q - \left(\frac{a}{q - r}\right) R$$

Graphic of two points Q and R in a line with a vector x between them. Q and R are shown with weights attached.

Decomposition in a 3-space

Decomposition of a vector in a vector 3-space

Suppose we have a vector 3-space represented by the trivector $\alpha \wedge \beta \wedge \gamma$. We wish to decompose a vector **x** in this 3-space to give one component in $\alpha \wedge \beta$ and the other in γ. Applying the decomposition formula gives:

$$x_{\alpha\wedge\beta} == \frac{(\alpha \wedge \beta) \vee (x \wedge \gamma)}{(\alpha \wedge \beta) \vee \gamma}$$

$$x_\gamma == \frac{\gamma \vee (x \wedge \alpha \wedge \beta)}{\gamma \vee (\alpha \wedge \beta)}$$

Graphic of two vectors α and β in a planar direction with a third vector γ in a third direction. A vector x is shown with two components, one in the planar direction and the other in the direction of γ.

Because $x \wedge \alpha \wedge \beta$ is a 3-element it can be seen immediately that the component x_γ can be written as a scalar multiple of γ where the scalar is expressed either as a ratio of regressive products (scalars) or exterior products (*n*-elements).

$$x_\gamma == \left(\frac{x \vee (\alpha \wedge \beta)}{\gamma \vee (\alpha \wedge \beta)}\right) \gamma == \left(\frac{\alpha \wedge \beta \wedge x}{\alpha \wedge \beta \wedge \gamma}\right) \gamma$$

The component $\mathbf{x}_{\alpha\wedge\beta}$ will be a linear combination of α and β. To show this we can expand the expression above for $\mathbf{x}_{\alpha\wedge\beta}$ using the Common Factor Axiom.

$$\mathbf{x}_{\alpha\wedge\beta} == \frac{(\alpha\wedge\beta)\vee(\mathbf{x}\wedge\gamma)}{(\alpha\wedge\beta)\vee\gamma} == \frac{(\alpha\wedge\mathbf{x}\wedge\gamma)\vee\beta}{(\alpha\wedge\beta)\vee\gamma} - \frac{(\beta\wedge\mathbf{x}\wedge\gamma)\vee\alpha}{(\alpha\wedge\beta)\vee\gamma}$$

Rearranging these two terms into a similar form as that derived for $\mathbf{x}_\gamma$ gives:

$$\mathbf{x}_{\alpha\wedge\beta} == \left(\frac{\mathbf{x}\wedge\beta\wedge\gamma}{\alpha\wedge\beta\wedge\gamma}\right)\alpha + \left(\frac{\alpha\wedge\mathbf{x}\wedge\gamma}{\alpha\wedge\beta\wedge\gamma}\right)\beta$$

Of course we could have obtained this decomposition directly by using the results of the decomposition formula [3.51] for decomposing a 1-element into a linear combination of the factors of an *n*-element.

$$\mathbf{x} == \left(\frac{\mathbf{x}\wedge\beta\wedge\gamma}{\alpha\wedge\beta\wedge\gamma}\right)\alpha + \left(\frac{\alpha\wedge\mathbf{x}\wedge\gamma}{\alpha\wedge\beta\wedge\gamma}\right)\beta + \left(\frac{\alpha\wedge\beta\wedge\mathbf{x}}{\alpha\wedge\beta\wedge\gamma}\right)\gamma$$

Decomposition of a point in a plane

Suppose we have a plane represented by the bound bivector $\mathbb{O}\wedge\mathbf{Q}\wedge\mathbf{R}$. We wish to decompose a point $\mathbf{P}$ in this plane to give one component in the line represented by $\mathbb{O}\wedge\mathbf{Q}$ and the other as a scalar multiple of $\mathbf{R}$. Applying the decomposition formula derived in the previous section gives:

$$\mathbf{P} == \left(\frac{\mathbf{P}\wedge\mathbf{Q}\wedge\mathbf{R}}{\mathbb{O}\wedge\mathbf{Q}\wedge\mathbf{R}}\right)\mathbb{O} + \left(\frac{\mathbb{O}\wedge\mathbf{P}\wedge\mathbf{R}}{\mathbb{O}\wedge\mathbf{Q}\wedge\mathbf{R}}\right)\mathbf{Q} + \left(\frac{\mathbb{O}\wedge\mathbf{Q}\wedge\mathbf{P}}{\mathbb{O}\wedge\mathbf{Q}\wedge\mathbf{R}}\right)\mathbf{R}$$

From this we can read off immediately the components $\mathbf{P}_{\mathbb{O}\wedge\mathbf{Q}}$ and $\mathbf{P}_\mathbf{R}$ we seek. $\mathbf{P}_{\mathbb{O}\wedge\mathbf{Q}}$ is the weighted point:

$$\mathbf{P}_{\mathbb{O}\wedge\mathbf{Q}} == \left(\frac{\mathbf{P}\wedge\mathbf{Q}\wedge\mathbf{R}}{\mathbb{O}\wedge\mathbf{Q}\wedge\mathbf{R}}\right)\left(\mathbb{O} + \left(\frac{\mathbb{O}\wedge\mathbf{P}\wedge\mathbf{R}}{\mathbf{P}\wedge\mathbf{Q}\wedge\mathbf{R}}\right)\mathbf{Q}\right)$$

whilst the weighted point located at $\mathbf{R}$ is:

$$\mathbf{P}_\mathbf{R} == \left(\frac{\mathbb{O}\wedge\mathbf{Q}\wedge\mathbf{P}}{\mathbb{O}\wedge\mathbf{Q}\wedge\mathbf{R}}\right)\mathbf{R}$$

Graphic of a line and points P and R external to it. A line joins R and P and intersects L.

Decomposition in a 4-space

Decomposition of a vector in a vector 4-space

Suppose we have a vector 4-space represented by the trivector $\alpha\wedge\beta\wedge\gamma\wedge\delta$. We wish to decompose a vector $\mathbf{x}$ in this 4-space. Applying the decomposition formula [3.51] gives:

$$\mathbf{x} == \left(\frac{\mathbf{x}\wedge\beta\wedge\gamma\wedge\delta}{\alpha\wedge\beta\wedge\gamma\wedge\delta}\right)\alpha + \left(\frac{\alpha\wedge\mathbf{x}\wedge\gamma\wedge\delta}{\alpha\wedge\beta\wedge\gamma\wedge\delta}\right)\beta + \left(\frac{\alpha\wedge\beta\wedge\mathbf{x}\wedge\delta}{\alpha\wedge\beta\wedge\gamma\wedge\delta}\right)\gamma + \left(\frac{\alpha\wedge\beta\wedge\gamma\wedge\mathbf{x}}{\alpha\wedge\beta\wedge\gamma\wedge\delta}\right)\delta$$

For simplicity we write $\mathbf{x}$ as:

$$\mathbf{x} == \mathbf{a}\,\alpha + \mathbf{b}\,\beta + \mathbf{c}\,\gamma + \mathbf{d}\,\delta$$

where the coefficients $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, and $\mathbf{d}$ are defined by the first equation above.

We can rearrange these components in whatever combinations we require. For example if we wanted the decomposition of $\mathbf{x}$ into three components, one parallel to α, one parallel to β, and one parallel to $\gamma \wedge \delta$ we would simply write:

$$\mathbf{x} == \mathbf{x}_\alpha + \mathbf{x}_\beta + \mathbf{x}_{\gamma\wedge\delta} == (\mathbf{a}\,\alpha) + (\mathbf{b}\,\beta) + (\mathbf{c}\,\gamma + \mathbf{d}\,\delta)$$

Of course, as we have seen in many of the examples above, decomposition of a point or vector in a 4-space whose elements are interpreted as points or vectors follows the same process: only the interpretation of the symbols differs.

Decomposition of a point or vector in an *n*-space

Decomposition of a point or vector in a space of *n* dimensions is generally most directly accomplished by using the decomposition formula [3.51] derived in Chapter 3: The Regressive Product. For example the decomposition of a point can be written:

$$\mathbf{P} == \sum_{i=1}^{n} \left(\frac{\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \mathbf{P} \wedge \cdots \wedge \alpha_n}{\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_i \wedge \cdots \wedge \alpha_n} \right) \alpha_i \qquad 4.5$$

As we have already seen for the 4-space example above, the components are simply arranged in whatever combinations are required to achieve the decomposition.

4.6 Geometrically Interpreted Spaces

Vector and point spaces

The space of a simple non-zero *m*-element $\underset{m}{\alpha}$ has been defined in Section 2.3 as the set of 1-elements $\mathbf{x}$ whose exterior product with $\underset{m}{\alpha}$ is zero: $\{\mathbf{x} : \underset{m}{\alpha} \wedge \mathbf{x} == 0\}$.

If $\mathbf{x}$ is interpreted as a vector $\mathbf{v}$, then the *vector space* of $\underset{m}{\alpha}$ is defined as $\{\mathbf{v} : \underset{m}{\alpha} \wedge \mathbf{v} == 0\}$.

If $\mathbf{x}$ is interpreted as a point $\mathbf{P}$, then the *point space* of $\underset{m}{\alpha}$ is defined as $\{\mathbf{P} : \underset{m}{\alpha} \wedge \mathbf{P} == 0\}$.

The vector space of a simple *m*-vector is an *m*-dimensional vector space. Conversely, the *m*-dimensional vector space may be said to be *defined* by the *m*-vector.

The point space of a bound simple *m*-vector is called an *m-plane* (sometimes *multiplane*). Thus the point space of a bound vector is a 1-plane (or line) and the point space of a bound simple bivector is a 2-plane (or, simply, a plane). The *m*-plane will be said to be *defined* by the bound simple *m*-vector.

The geometric interpretation for the notion of set inclusion is taken as 'to lie in'. Thus for example, a point may be said *to lie in* an *m*-plane.

The point and vector spaces for a bound simple *m*-element are tabulated below.

m	$\mathbf{P} \wedge \underset{m}{\alpha}$	**Point space**	**Vector space**
0	bound scalar	point	
1	bound vector	line	1 –dimensional vector space
2	bound simple bivector	plane	2 –dimensional vector space
m	bound simple *m*–vector	*m*–plane	m –dimensional vector space
$n-1$	bound $(n-1)$ –vector	hyperplane	$(n-1)$ –dimensional vector space
n	bound *n*–vector	*n*–plane	*n* –dimensional vector space

Two congruent bound simple *m*-vectors $\mathbf{P} \wedge \underset{m}{\alpha}$ and $\mathbf{a}\, \mathbf{P} \wedge \underset{m}{\alpha}$ define the same *m*-plane. Thus, for example, the point $\mathbb{O}+\mathbf{x}$ and the weighted point $2\mathbb{O}+2\mathbf{x}$ define the same point.

Bound simple *m*-vectors as *m*-planes

We have defined an *m*-plane as a set of points *defined by* a bound simple *m*-vector. It will often turn out however to be more convenient and conceptually fruitful to work with *m*-planes *as if* they were the bound simple *m*-vectors which define them. This is in fact the approach taken by Grassmann and the early workers in the Grassmannian tradition (for example, [Whitehead], [Hyde] and [Forder]). This will be satisfactory provided that the equality relationship we define for *m*-planes is that of *congruence* rather than the more specific *equals*.

Thus we may, when speaking in a geometric context, refer to a bound simple *m*-vector as an *m*-plane and *vice versa*. Hence in saying Π is an *m*-plane, we are also saying that all $\mathbf{a}\,\Pi$ (where $\mathbf{a}$ is a scalar factor, not zero) is the same *m*-plane.

Coordinate spaces

The *coordinate spaces* of a Grassmann algebra are the spaces defined by the basis elements.

The *coordinate m-spaces* are the spaces defined by the basis elements of $\underset{m}{\Lambda}$. For example, if $\underset{1}{\Lambda}$ has basis $\{\mathbb{O},\ \mathbf{e}_1,\ \mathbf{e}_2,\ \mathbf{e}_3\}$, that is, we are working in the bound vector 3-space $\mathbb{P}_3$, then the Grassmann algebra it generates has basis:

```
ℙ3; BasisΛ[]
```

$\{1,\ \mathbb{O},\ e_1,\ e_2,\ e_3,\ \mathbb{O}\wedge e_1,\ \mathbb{O}\wedge e_2,\ \mathbb{O}\wedge e_3,\ e_1\wedge e_2,\ e_1\wedge e_3,\ e_2\wedge e_3,\ \mathbb{O}\wedge e_1\wedge e_2,\ \mathbb{O}\wedge e_1\wedge e_3,\ \mathbb{O}\wedge e_2\wedge e_3,\ e_1\wedge e_2\wedge e_3,\ \mathbb{O}\wedge e_1\wedge e_2\wedge e_3\}$

Each one of these basis elements defines a coordinate space. Most familiar are the coordinate *m*-planes. The coordinate 1-planes $\mathbb{O}\wedge\mathbf{e}_1$, $\mathbb{O}\wedge\mathbf{e}_2$, $\mathbb{O}\wedge\mathbf{e}_3$ define the coordinate axes, while the coordinate 2-planes $\mathbb{O}\wedge\mathbf{e}_1\wedge\mathbf{e}_2$, $\mathbb{O}\wedge\mathbf{e}_1\wedge\mathbf{e}_3$, $\mathbb{O}\wedge\mathbf{e}_2\wedge\mathbf{e}_3$ define the coordinate planes. Additionally however there are the coordinate vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ and the coordinate bivectors $\mathbf{e}_1\wedge\mathbf{e}_2$, $\mathbf{e}_1\wedge\mathbf{e}_3$, $\mathbf{e}_2\wedge\mathbf{e}_3$.

Perhaps less familiar is the fact that there are no coordinate *m*-planes in a vector space, but rather simply coordinate *m*-vectors.

Geometric dependence

In Chapter 2 the notion of dependence was discussed for elements of a linear space. Non-zero 1-elements are said to be dependent if and only if their exterior product is zero.

If the elements concerned have been endowed with a geometric interpretation, the notion of dependence takes on an additional geometric interpretation, as the following table shows.

$x_1 \wedge x_2 == 0$ $P_1 \wedge P_2 == 0$	x_1, x_2 are parallel (co–directional) P_1, P_2 are coincident
$x_1 \wedge x_2 \wedge x_3 == 0$ $P_1 \wedge P_2 \wedge P_3 == 0$	x_1, x_2, x_3 are co–2–directional (or parallel) P_1, P_2, P_3 are collinear (or coincident)
$x_1 \wedge \cdots \wedge x_m == 0$ $P_1 \wedge \cdots \wedge P_m == 0$	$x_1, \cdots, x_m$ are co–k–directional, $k < m$ $P_1, \cdots, P_m$ are co–k–planar, $k < m - 1$

Geometric duality

The concept of duality introduced in Chapter 3 is most striking when interpreted geometrically. Suppose:

P defines a point
L defines a line
π defines a plane
V defines a 3-plane

In what follows we tabulate the dual relationships of these entities to each other.

Duality in a plane

In a plane there are just three types of geometric entity: points, lines and planes. In the table below we can see that in the plane, points and lines are 'dual' entities, and planes and scalars are 'dual' entities, because their definitions convert under the application of the Duality Principle.

$L \equiv P_1 \wedge P_2$	$P \equiv L_1 \vee L_2$
$\pi \equiv P_1 \wedge P_2 \wedge P_3$	$1 \equiv L_1 \vee L_2 \vee L_3$
$\pi \equiv L \wedge P$	$1 \equiv P \vee L$

Duality in a 3-plane

In the 3-plane there are just four types of geometric entity: points, lines, planes and 3-planes. In the table below we can see that in the 3-plane, lines are self-dual, points and planes are now dual, and scalars are now dual to 3-planes.

$L \equiv P_1 \wedge P_2$	$L \equiv \pi_1 \vee \pi_2$
$\pi \equiv P_1 \wedge P_2 \wedge P_3$	$P \equiv \pi_1 \vee \pi_2 \vee \pi_3$
$V \equiv P_1 \wedge P_2 \wedge P_3 \wedge P_4$	$1 \equiv \pi_1 \vee \pi_2 \vee \pi_3 \vee \pi_4$
$\pi \equiv L \wedge P$	$P \equiv L \vee \pi$
$V \equiv L_1 \wedge L_2$	$1 \equiv L_1 \vee L_2$
$V \equiv \pi \wedge P$	$1 \equiv P \vee \pi$

Duality in an *n*-plane

From these cases the types of relationships in higher dimensions may be composed straightforwardly. For example, if **P** defines a point and **H** defines a hyperplane ((*n*–1)-plane), then we have the dual formulations:

$H \equiv P_1 \wedge P_2 \wedge \cdots \wedge P_{n-1}$	$P \equiv H_1 \vee H_2 \vee \cdots \vee H_{n-1}$

4.7 *m*-planes

In the previous section *m*-planes have been defined as point spaces of bound simple *m*-vectors. In this section *m*-planes will be considered from three other aspects: the first in terms of a simple exterior product of points, the second as an *m*-vector and the third as an exterior quotient.

m-planes defined by points

Grassmann and those who wrote in the style of the *Ausdehnungslehre* considered the point more fundamental than the vector for exploring geometry. This approach indeed has its merits. An *m*-plane is quite straightforwardly defined and expressed as the (space of the) exterior product of *m*+1 points.

$$\Pi == P_0 \wedge P_1 \wedge P_2 \wedge \cdots \wedge P_m$$

m-planes defined by *m*-vectors

Consider a bound simple *m*-vector $P_0 \wedge x_1 \wedge x_2 \wedge \cdots \wedge x_m$. Its *m*-plane is the set of points **P** such that:

$$P \wedge P_0 \wedge x_1 \wedge x_2 \wedge \cdots \wedge x_m == 0$$

This equation is equivalent to the statement: *there exist scalars* a, a_0, a_i, *not all zero, such that*:

$$a\, P + a_0\, P_0 + \Sigma\, a_i\, x_i == 0$$

And since this is only possible if $a == -a_0$ (since for the sum to be zero, it must be a sum of vectors) then:

$$a\, (P - P_0) + \Sigma\, a_i\, x_i == 0$$

or equivalently:

$$(P - P_0) \wedge x_1 \wedge x_2 \wedge \cdots \wedge x_m == 0$$

We are thus lead to the following alternative definition of an *m*-plane: an *m*-plane defined by the bound simple *m*-vector $P_0 \wedge x_1 \wedge x_2 \wedge \cdots \wedge x_m$ is the set of points:

$$\{P : (P - P_0) \wedge x_1 \wedge x_2 \wedge \cdots \wedge x_m == 0\}$$

This is of course equivalent to the usual definition of an *m*-plane. That is, since the vectors $(P - P_0)$, x_1, x_2, $\cdots$, x_m are dependent, then for scalar parameters t_i:

$$(P - P_0) == t_1 x_1 + t_2 x_2 + \cdots + t_m x_m \qquad 4.6$$

m-planes as exterior quotients

The alternative definition of an *m*-plane developed above shows that an *m*-plane may be defined as the set of points P such that:

$$P \wedge x_1 \wedge x_2 \wedge \cdots \wedge x_m == P_0 \wedge x_1 \wedge x_2 \wedge \cdots \wedge x_m$$

'Solving' for P and noting from Section 2.11 that the quotient of an (*m*+1)-element by an *m*-element contained in it is a 1-element with *m* arbitrary scalar parameters, we can write:

$$P == \frac{P_0 \wedge x_1 \wedge x_2 \wedge \cdots \wedge x_m}{x_1 \wedge x_2 \wedge \cdots \wedge x_m} == P_0 + t_1 x_1 + t_2 x_2 + \cdots + t_m x_m$$

$$P == \frac{P_0 \wedge x_1 \wedge x_2 \wedge \cdots \wedge x_m}{x_1 \wedge x_2 \wedge \cdots \wedge x_m} \qquad 4.7$$

The operator ∂

We can define an operator ∂ which takes a simple (*m*+1)-element of the form $P_0 \wedge P_1 \wedge P_2 \wedge \cdots \wedge P_m$ and converts it to an *m*-element of the form $(P_1 - P_0) \wedge (P_2 - P_0) \wedge \cdots \wedge (P_m - P_0)$.

The interesting property of this operation is that when it is applied twice, the result is zero. Operationally, $\partial^2 == 0$. For example:

$$\partial(P) == 1$$

$$\partial(P_0 \wedge P_1) == P_1 - P_0$$
$$\partial(P_1 - P_0) == 1 - 1 == 0$$

$$\partial(P_0 \wedge P_1 \wedge P_2) == P_1 \wedge P_2 + P_2 \wedge P_0 + P_0 \wedge P_1$$
$$\partial(P_1 \wedge P_2 + P_2 \wedge P_0 + P_0 \wedge P_1) == P_2 - P_1 + P_0 - P_2 + P_1 - P_0 == 0$$

Remember that $P_1 \wedge P_2 + P_2 \wedge P_0 + P_0 \wedge P_1$ is simple since it may be expressed as

$$(P_1 - P_0) \wedge (P_2 - P_0) == (P_2 - P_1) \wedge (P_0 - P_1) == (P_0 - P_2) \wedge (P_1 - P_2)$$

This property of nilpotence is shared by the boundary operator of algebraic topology and the exterior derivative. Furthermore, if a product with a given 1-element is considered an operation, then the exterior, regressive and interior products are all likewise nilpotent.

4.8 Line Coordinates

We have already seen that lines are defined by bound vectors independent of the dimension of the space. We now look at the types of coordinate descriptions we can use to define lines in bound spaces (multiplanes) of various dimensions.

For simplicity of exposition we refer to a bound vector as 'a line', rather than as 'defining a line'.

Lines in a plane

To explore lines in a plane, we first declare the basis of the plane: $\mathbb{P}_2$.

$$\mathbb{P}_2$$

$$\{\mathbb{O}, e_1, e_2\}$$

A line in a plane can be written in several forms. The most intuitive form perhaps is as a product of two points **𝕆+x** and **𝕆+y** where **x** and **y** are position vectors.

$$L \equiv (\mathbb{O} + x) \wedge (\mathbb{O} + y)$$

Graphic of a line through two points specified by position vectors.

We can automatically generate a basis form for each of the position vectors **x** and **y** by using the *GrassmannAlgebra* `CreateVector` function.

```
{X = CreateVector[x], Y = CreateVector[y]}
```

$$\{e_1 x_1 + e_2 x_2, e_1 y_1 + e_2 y_2\}$$

$$L \equiv (\mathbb{O} + X) \wedge (\mathbb{O} + Y)$$

$$L \equiv (\mathbb{O} + e_1 x_1 + e_2 x_2) \wedge (\mathbb{O} + e_1 y_1 + e_2 y_2)$$

Or, we can express the line as the product of any point in it and a vector parallel to it. For example:

```
L ≡ (𝕆 + X) ∧ (Y - X) ≡ (𝕆 + Y) ∧ (Y - X) // Simplify
```

$$L \equiv (\mathbb{O} + e_1 x_1 + e_2 x_2) \wedge (e_1 (-x_1 + y_1) + e_2 (-x_2 + y_2)) \equiv (\mathbb{O} + e_1 y_1 + e_2 y_2) \wedge (e_1 (-x_1 + y_1) + e_2 (-x_2 + y_2))$$

Graphic of a line through a point parallel to the difference between two position vectors.

Alternatively, we can express $\mathbf{L}$ without specific reference to points in it. For example:

$$\mathbf{L} \equiv \mathbb{O} \wedge (a\, e_1 + b\, e_2) + c\, e_1 \wedge e_2$$

The first term $\mathbb{O} \wedge (a\, e_1 + b\, e_2)$ is a vector bound through the origin, and hence defines a line through the origin. The second term $c\, e_1 \wedge e_2$ is a bivector whose addition represents a shift in the line parallel to itself, away from the origin.

Graphic showing a line through the origin plus a shift due to the addition of a bivector. Don't draw the basis vectors at right angles.

We know that this can indeed represent a line since we can factorize it into any of the forms:

- A line of gradient $\frac{b}{a}$ through the point with coordinate $\frac{c}{b}$ on the $\mathbb{O} \wedge e_1$ axis.

$$\mathbf{L} \equiv \left(\mathbb{O} + \frac{c}{b} e_1\right) \wedge (a\, e_1 + b\, e_2)$$

Graphic of a line through a point on the axis with given gradient.

- A line of gradient $\frac{b}{a}$ through the point with coordinate $-\frac{c}{a}$ on the $\mathbb{O} \wedge e_2$ axis.

$$\mathbf{L} \equiv \left(\mathbb{O} - \frac{c}{a} e_2\right) \wedge (a\, e_1 + b\, e_2)$$

Graphic of a line through a point on the axis with given gradient.

- Or, a line through both points.

$$\mathbf{L} \equiv \frac{a\, b}{c} \left(\mathbb{O} - \frac{c}{a} e_2\right) \wedge \left(\mathbb{O} + \frac{c}{b} e_1\right)$$

Of course the scalar factor $\frac{a\, b}{c}$ is inessential so we can just as well say:

$$\mathbf{L} \equiv \left(\mathbb{O} - \frac{c}{a} e_2\right) \wedge \left(\mathbb{O} + \frac{c}{b} e_1\right)$$

Graphic of a line through a point on each axis.

◆ Information required to express a line in a plane

The expression of the line above in terms of a pair of points requires the four coordinates of the points. Expressed without specific reference to points, we seem to need three parameters. However, the last expression shows, as expected, that it is really only two parameters that are necessary (*viz* $y = m\, x + c$).

Lines in a 3-plane

Lines in a 3-plane $\mathbb{P}_3$ have the same form when expressed in coordinate-free notation as they do in a plane $\mathbb{P}_2$. Remember that a 3-plane is a bound vector 3-space whose basis may be chosen as 3 independent vectors and a point, or equivalently as 4 independent points. For example, we can still express a line in a 3-plane in any of the following equivalent forms.

$$L \equiv (\mathbb{O} + x) \wedge (\mathbb{O} + y)$$
$$L \equiv (\mathbb{O} + x) \wedge (y - x)$$
$$L \equiv (\mathbb{O} + y) \wedge (y - x)$$
$$L \equiv \mathbb{O} \wedge (y - x) + x \wedge y$$

Here, **x** and **y** are independent vectors in the 3-plane.

The coordinate form however will appear somewhat different to that in the 2-plane case. To explore this, we redeclare the basis as $\mathbb{P}_3$.

$\mathbb{P}_3$

$\{\mathbb{O}, e_1, e_2, e_3\}$

{X = CreateVector[x], Y = CreateVector[y]}

$\{e_1\, x_1 + e_2\, x_2 + e_3\, x_3,\ e_1\, y_1 + e_2\, y_2 + e_3\, y_3\}$

$L \equiv (\mathbb{O} + X) \wedge (\mathbb{O} + Y)$

$L \equiv (\mathbb{O} + e_1\, x_1 + e_2\, x_2 + e_3\, x_3) \wedge (\mathbb{O} + e_1\, y_1 + e_2\, y_2 + e_3\, y_3)$

Multiplying out this expression gives:

$L \equiv \mathcal{G}[(\mathbb{O} + e_1\, x_1 + e_2\, x_2 + e_3\, x_3) \wedge (\mathbb{O} + e_1\, y_1 + e_2\, y_2 + e_3\, y_3)]$

$$L \equiv \mathbb{O} \wedge (e_1\,(-x_1 + y_1) + e_2\,(-x_2 + y_2) + e_3\,(-x_3 + y_3)) + (-x_2\, y_1 + x_1\, y_2)\, e_1 \wedge e_2 + (-x_3\, y_1 + x_1\, y_3)\, e_1 \wedge e_3 + (-x_3\, y_2 + x_2\, y_3)\, e_2 \wedge e_3$$

The scalar coefficients in this expression are sometimes called the *Plücker coordinates* of the line.

Alternatively, we can express **L** in terms of basis elements, but without specific reference to points or vectors in it. For example:

$$L \equiv \mathbb{O} \wedge (a\, e_1 + b\, e_2 + c\, e_3) + d\, e_1 \wedge e_2 + e\, e_2 \wedge e_3 + f\, e_1 \wedge e_3$$

The first term $\mathbb{O} \wedge (a\, e_1 + b\, e_2 + c\, e_3)$ is a vector bound through the origin, and hence defines a line through the origin. The second term $d\, e_1 \wedge e_2 + e\, e_2 \wedge e_3 + f\, e_1 \wedge e_3$ is a bivector whose addition represents a shift in the line parallel to itself, away from the origin. In order to effect this shift, however, it is necessary that the bivector contain the vector $(a\, e_1 + b\, e_2 + c\, e_3)$. Hence there will be some constraint on the coefficients **d**, **e**, and **f**. To determine this we only need to determine the condition that the exterior product of the vector and the bivector is zero.

$$\mathcal{G}[(a\, e_1 + b\, e_2 + c\, e_3) \wedge (d\, e_1 \wedge e_2 + e\, e_2 \wedge e_3 + f\, e_1 \wedge e_3) == 0]$$

$$(c\, d + a\, e - b\, f)\; e_1 \wedge e_2 \wedge e_3 == 0$$

Alternatively, this constraint amongst the coefficients could have been obtained by noting that in order to be a line, **L** must be simple, hence the exterior product with itself must be zero.

$$L = \mathbb{O} \wedge (a\, e_1 + b\, e_2 + c\, e_3) + d\, e_1 \wedge e_2 + e\, e_2 \wedge e_3 + f\, e_1 \wedge e_3;$$

$$\mathcal{G}[L \wedge L == 0]$$

$$2\; (c\, d + a\, e - b\, f)\; \mathbb{O} \wedge e_1 \wedge e_2 \wedge e_3 == 0$$

Thus the constraint that the coefficients must obey in order for a general bound vector of the form **L** to be a line in a 3-plane is that:

$$c\, d + a\, e - b\, f == 0$$

This constraint is sometimes referred to as the *Plücker identity*.

Given this constraint, and supposing neither **a**, **b** or **c** is zero, we can factorize the line into any of the following forms:

$$L == \left(\mathbb{O} + \frac{f}{c}\, e_1 + \frac{e}{c}\, e_2\right) \wedge (a\, e_1 + b\, e_2 + c\, e_3)$$

$$L == \left(\mathbb{O} - \frac{d}{a}\, e_2 - \frac{f}{a}\, e_3\right) \wedge (a\, e_1 + b\, e_2 + c\, e_3)$$

$$L == \left(\mathbb{O} + \frac{d}{b}\, e_1 - \frac{e}{b}\, e_3\right) \wedge (a\, e_1 + b\, e_2 + c\, e_3)$$

Each of these forms represents a line in the direction of $a\, e_1 + b\, e_2 + c\, e_3$ and intersecting a coordinate plane. For example, the first form intersects the $\mathbb{O} \wedge e_1 \wedge e_2$ coordinate plane in the point $\mathbb{O} + \frac{f}{c}\, e_1 + \frac{e}{c}\, e_2$ with coordinates ($\frac{f}{c}$, $\frac{e}{c}$, **0**).

The most compact form, in terms of the number of scalar parameters used, is when **L** is expressed as the product of two points, each of which lies in a coordinate plane.

$$L = \frac{a\, c}{f} \left(\mathbb{O} - \frac{d}{a}\, e_2 - \frac{f}{a}\, e_3\right) \wedge \left(\mathbb{O} + \frac{f}{c}\, e_1 + \frac{e}{c}\, e_2\right);$$

We can verify that this formulation gives us the original form of the line by expanding the product and substituting the constraint relation previously obtained.

$$\mathcal{G}[L]\; /.\; (c\, d + a\, e \to f\, b)$$

$$\mathbb{O} \wedge (a\, e_1 + b\, e_2 + c\, e_3) + d\, e_1 \wedge e_2 + f\, e_1 \wedge e_3 + e\, e_2 \wedge e_3$$

Similar expressions may be obtained for **L** in terms of points lying in the other coordinate planes. To summarize, there are three possibilities in a 3-plane, corresponding to the number of different pairs of coordinate 2-planes in the 3-plane.

$$\mathbf{L} \equiv (\mathbb{O} + x_1\, e_1 + x_2\, e_2) \wedge (\mathbb{O} + y_2\, e_2 + y_3\, e_3)$$
$$\mathbf{L} \equiv (\mathbb{O} + x_1\, e_1 + x_2\, e_2) \wedge (\mathbb{O} + z_1\, e_1 + z_3\, e_3) \qquad 4.8$$
$$\mathbf{L} \equiv (\mathbb{O} + y_2\, e_2 + y_3\, e_3) \wedge (\mathbb{O} + z_1\, e_1 + z_3\, e_3)$$

Graphic of a line through 3 points, one in each coordinate plane.

Information required to express a line in a 3-plane

As with a line in a 2-plane, we find that a line in a 3-plane is expressed with the minimum number of parameters by expressing it as the product of two points, each in one of the coordinate planes. In this form, there are just 4 independent scalar parameters (coordinates) required to express the line.

Checking the invariance of the description of the line

We can use *GrassmannAlgebra* to explore the invariance of how a line is expressed. Again, suppose we are in a 3-plane.

```
ℙ3
{𝕆, e1, e2, e3}
```

Define a line **L** as the exterior product of two points, one in the $\mathbb{O} \wedge e_1 \wedge e_2$ coordinate plane and one in the $\mathbb{O} \wedge e_2 \wedge e_3$ coordinate plane.

```
L = (𝕆 + x1 e1 + x2 e2) ∧ (𝕆 + y2 e2 + y3 e3);
```

Declare the coordinates as scalars.

```
DeclareExtraScalars[{x1, x2, y2, y3}];
```

Verify that the intersection of the line with the first coordinate plane does indeed give a point congruent to the first point.

```
P1 = 𝒢[L ∨ (𝕆 ∧ e1 ∧ e2)] // ToCongruenceForm
```

$$\frac{(\mathbb{O} + e_1\, x_1 + e_2\, x_2)\, y_3}{\mathbb{k}}$$

Next determine the (weighted) point in the line in the *third* coordinate plane (the coordinate plane which did not figure in the original definition of the line).

```
P2 = 𝒢[L ∨ (𝕆 ∧ e1 ∧ e3)] // ToCongruenceForm
```

$$\frac{\mathbb{O}\,(x_2 - y_2) - e_1\, x_1\, y_2 + e_3\, x_2\, y_3}{\mathbb{k}}$$

We can now confirm that the exterior product of these two points is still congruent to the original specification of the line by expanding the quotient of the two expressions and showing that it reduces to a scalar.

```
ScalarQ[Simplify[FactorScalars[ExpandProducts[𝒢[P1 ∧ P2] / 𝒢[L]]]]]
```

```
True
```

Lines in a 4-plane

Lines in a 4-plane $\mathbb{P}_4$ have the same form when expressed in coordinate-free notation as they do in any multiplane.

To obtain the Plücker coordinates of a line in a 4-plane, express the line as the exterior product of two points and multiply it out. The resulting coefficients of the basis elements are the Plücker coordinates of the line.

Additionally, from the results above, we can expect that a line in a 4-plane may be expressed with the least number of scalar parameters as the exterior product of two points, each point lying in one of the coordinate 3-planes. For example, the expression for the line as the product of the points in the coordinate 3-planes $\mathbb{O} \wedge e_1 \wedge e_2 \wedge e_3$ and $\mathbb{O} \wedge e_2 \wedge e_3 \wedge e_4$ is

$$L = (\mathbb{O} + x_1\, e_1 + x_2\, e_2 + x_3\, e_3) \wedge (\mathbb{O} + y_2\, e_2 + y_3\, e_3 + y_4\, e_4)$$

Lines in an *m*-plane

The formulae below summarize some of the expressions for defining a line, valid in a multiplane of any dimension.

Coordinate-free expressions may take any of a number of forms. For example:

$$\begin{aligned} L &\equiv (\mathbb{O} + x) \wedge (\mathbb{O} + y) \\ L &\equiv (\mathbb{O} + x) \wedge (y - x) \\ L &\equiv (\mathbb{O} + y) \wedge (y - x) \\ L &\equiv \mathbb{O} \wedge (y - x) + x \wedge y \end{aligned} \qquad 4.9$$

A line can be expressed in terms of the 2*m* coordinates of any two points on it.

$$L \equiv (\mathbb{O} + x_1\, e_1 + x_2\, e_2 + \cdots + x_m\, e_m) \wedge (\mathbb{O} + y_1\, e_1 + y_2\, e_2 + \cdots + y_m\, e_m) \qquad 4.10$$

When multiplied out, the expression for the line takes a form explicitly displaying the Plücker coordinates of the line.

$$\begin{aligned} L \equiv\ & \mathbb{O} \wedge ((y_1 - x_1)\, e_1 + (y_2 - x_2)\, e_2 + \cdots + (y_m - x_m)\, e_m) \\ & + (x_1\, y_2 - x_2\, y_1)\, e_1 \wedge e_2 + (x_1\, y_3 - x_3\, y_1)\, e_1 \wedge e_3 + \\ & (x_1\, y_4 - x_4\, y_1)\, e_1 \wedge e_4 + \cdots + (x_{m-1}\, y_m - x_m\, y_{m-1})\, e_{m-1} \wedge e_m \end{aligned} \qquad 4.11$$

Alternatively, a line in an *m*-plane $\mathbb{O} \wedge \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \cdots \wedge \mathbf{e}_m$ can be expressed in terms of its intersections with two of its coordinate (*m*–1)-planes, $\mathbb{O} \wedge \mathbf{e}_1 \wedge \cdots \wedge \square_i \wedge \cdots \wedge \mathbf{e}_m$ and $\mathbb{O} \wedge \mathbf{e}_1 \wedge \cdots \wedge \square_j \wedge \cdots \wedge \mathbf{e}_m$ say. The notation $\square_i$ means that the *i*th element or term is missing.

$$L \equiv (\mathbb{O} + x_1\, e_1 + \cdots + \square_i + \cdots + x_m\, e_m) \wedge (\mathbb{O} + y_1\, e_1 + \cdots + \square_j + \cdots + y_m\, e_m) \qquad 4.12$$

This formulation indicates that a line in *m*-space has at most 2(*m*–1) independent parameters required to describe it.

It also implies that in the special case when the line lies in one of the coordinate (*m*–1)-spaces, it can be even more economically expressed as the product of two points, each lying in one of the coordinate (*m*–2)-spaces contained in the (m–1)-space. And so on.

4.9 Plane Coordinates

We have already seen that planes are defined by simple bound bivectors independent of the dimension of the space. We now look at the types of coordinate descriptions we can use to define planes in bound spaces (multiplanes) of various dimensions.

Planes in a 3-plane

A plane Π in a 3-plane can be written in several forms. The most intuitive form perhaps is as a product of three non-collinear points $\mathbb{O}+x$, $\mathbb{O}+y$ and $\mathbb{O}+z$, where x, y and z are vectors.

$$\Pi \equiv (\mathbb{O} + x) \wedge (\mathbb{O} + y) \wedge (\mathbb{O} + z)$$

Graphic of a plane with the preceding definition.

Or, we can express it as the product of any two different points in it and a vector parallel to it (but not in the direction of the line joining the two points). For example:

$$\Pi \equiv (\mathbb{O} + x) \wedge (\mathbb{O} + y) \wedge (z - x)$$

Graphic of a plane with the preceding definition.

Or, we can express it as the product of any point in it and any two independent vectors parallel to it. For example:

$$\Pi \equiv (\mathbb{O} + x) \wedge (y - x) \wedge (z - x)$$

Graphic of a plane with the preceding definition.

Or, we can express it as the product of any line in it and any point in it not in the line. For example:

$$\Pi \equiv \mathbf{L} \wedge (\mathbb{O} + \mathbf{x})$$

Graphic of a plane with the preceding definition.

Or, we can express it as the product of any line in it and any vector parallel to it (but not parallel to the line). For example:

$$\Pi \equiv \mathbf{L} \wedge (\mathbf{z} - \mathbf{x})$$

Graphic of a plane with the preceding definition.

Given a basis, we can always express the plane in terms of the coordinates of the points or vectors in the expressions above. However the form which requires the least number of coordinates is that which expresses the plane as the exterior product of its three points of intersection with the coordinate axes.

$$\Pi \equiv (\mathbb{O} + \mathbf{a}\, \mathbf{e}_1) \wedge (\mathbb{O} + \mathbf{b}\, \mathbf{e}_2) \wedge (\mathbb{O} + \mathbf{c}\, \mathbf{e}_3)$$

Graphic of a plane with the preceding definition.

If the plane is parallel to one of the coordinate axes, say $\mathbb{O} \wedge \mathbf{e}_3$, it may be expressed as:

$$\Pi \equiv (\mathbb{O} + \mathbf{a}\, \mathbf{e}_1) \wedge (\mathbb{O} + \mathbf{b}\, \mathbf{e}_2) \wedge \mathbf{e}_3$$

Whereas, if it is parallel to two of the coordinate axes, say $\mathbb{O} \wedge \mathbf{e}_2$ and $\mathbb{O} \wedge \mathbf{e}_3$, it may be expressed as:

$$\Pi \equiv (\mathbb{O} + \mathbf{a}\, \mathbf{e}_1) \wedge \mathbf{e}_2 \wedge \mathbf{e}_3$$

If we wish to express a plane as the exterior product of its intersection points with the coordinate axes, we first determine its points of intersection with the axes and then take the exterior product of the resulting points. This leads to the following identity:

$$\Pi \equiv (\Pi \vee (\mathbb{O} \wedge \mathbf{e}_1)) \wedge (\Pi \vee (\mathbb{O} \wedge \mathbf{e}_2)) \wedge (\Pi \vee (\mathbb{O} \wedge \mathbf{e}_3))$$

Example: To express a plane in terms of its intersections with the coordinate axes

Suppose we have a plane in a 3-plane defined by three points.

$$\mathbb{P}_3;\ \Pi = (\mathbb{O} + \mathbf{e}_1 + 2\, \mathbf{e}_2 + 5\, \mathbf{e}_3) \wedge (\mathbb{O} - \mathbf{e}_1 + 9\, \mathbf{e}_2) \wedge (\mathbb{O} - 7\, \mathbf{e}_1 + 6\, \mathbf{e}_2 + 4\, \mathbf{e}_3)$$

$$(\mathbb{O} + e_1 + 2\, e_2 + 5\, e_3) \wedge (\mathbb{O} - e_1 + 9\, e_2) \wedge (\mathbb{O} - 7\, e_1 + 6\, e_2 + 4\, e_3)$$

To express this plane in terms of its intersections with the coordinate axes we calculate the intersection points with the axes.

$$\mathcal{G}[\Pi \vee (\mathbb{O} \wedge \mathbf{e}_1)]$$

$$\mathbb{O} \wedge e_1 \wedge e_2 \wedge e_3 \vee (13\, \mathbb{O} + 329\, e_1)$$

$$\mathcal{G}[\Pi \vee (\mathbb{O} \wedge e_2)]$$

$$\mathbb{O} \wedge e_1 \wedge e_2 \wedge e_3 \vee (38\,\mathbb{O} + 329\, e_2)$$

$$\mathcal{G}[\Pi \vee (\mathbb{O} \wedge e_3)]$$

$$\mathbb{O} \wedge e_1 \wedge e_2 \wedge e_3 \vee (48\,\mathbb{O} + 329\, e_3)$$

We then take the product of these points (ignoring the weights) to form the plane.

$$\Pi \equiv \left(\mathbb{O} + \frac{329\, e_1}{13}\right) \wedge \left(\mathbb{O} + \frac{329\, e_2}{38}\right) \wedge \left(\mathbb{O} + \frac{329\, e_3}{48}\right)$$

To verify that this is indeed the same plane, we can check to see if these points are in the original plane. For example:

$$\mathcal{G}\left[\Pi \wedge \left(\mathbb{O} + \frac{329\, e_1}{13}\right)\right]$$

$$0$$

Planes in a 4-plane

From the results above, we can expect that a plane in a 4-plane is most economically expressed as the product of three points, each point lying in one of the coordinate 2-planes. For example:

$$\Pi \equiv (\mathbb{O} + x_1\, e_1 + x_2\, e_2) \wedge (\mathbb{O} + y_2\, e_2 + y_3\, e_3) \wedge (\mathbb{O} + z_3\, e_3 + z_4\, e_4)$$

If a plane is expressed in any other form, we can express it in the form above by first determining its points of intersection with the coordinate planes and then taking the exterior product of the resulting points. This leads to the following identity:

$$\Pi \equiv (\Pi \vee (\mathbb{O} \wedge e_1 \wedge e_2)) \wedge (\Pi \vee (\mathbb{O} \wedge e_2 \wedge e_3)) \wedge (\Pi \vee (\mathbb{O} \wedge e_3 \wedge e_4))$$

Planes in an *m*-plane

A plane in an *m*-plane is most economically expressed as the product of three points, each point lying in one of the coordinate (*m*–2)-planes.

$$\Pi \equiv \left(\mathbb{O} + x_1\, e_1 + \cdots + \underbrace{x_{i_1}\, e_{i_1}} + \cdots + \underbrace{x_{i_2}\, e_{i_2}} + \cdots + x_m\, e_m\right)$$
$$\wedge \left(\mathbb{O} + y_1\, e_1 + \cdots + \underbrace{y_{i_3}\, e_{i_3}} + \cdots + \underbrace{y_{i_4}\, e_{i_4}} + \cdots + y_m\, e_m\right)$$
$$\wedge \left(\mathbb{O} + z_1\, e_1 + \cdots + \underbrace{z_{i_5}\, e_{i_5}} + \cdots + \underbrace{z_{i_6}\, e_{i_6}} + \cdots + z_m\, e_m\right)$$

Here the notation $\underbrace{x_i\, e_i}$ means that the term is *missing* from the sum.

This formulation indicates that a plane in an *m*-plane has at most 3(*m*–2) independent scalar parameters required to describe it.

4.10 Calculation of Intersections

The intersection of two lines in a plane

Suppose we wish to find the point **P** of intersection of two lines L_1 and L_2 in a plane. We have seen in the previous section how we could express a line in a plane as the exterior product of two points, and that these points could be taken as the points of intersection of the line with the coordinate axes.

First declare the basis of the plane, and then define the lines.

$\mathbb{P}_2$

$\{\mathbb{O}, e_1, e_2\}$

$L_1 = (\mathbb{O} + a\, e_1) \wedge (\mathbb{O} + b\, e_2);$
$L_2 = (\mathbb{O} + c\, e_1) \wedge (\mathbb{O} + d\, e_2);$

Next, take the regressive product of the two lines and simplify it.

$\mathcal{G}[L_1 \vee L_2]$

$\mathbb{O} \wedge e_1 \wedge e_2 \vee ((b\, c - a\, d)\, \mathbb{O} + a\, c\, (b - d)\, e_1 + b\, (-a + c)\, d\, e_2)$

This shows that the intersection point **P** is given by:

$$P = \mathbb{O} + \frac{a\, c\, (b - d)}{b\, c - a\, d}\, e_1 + \frac{b\, d\, (c - a)}{b\, c - a\, d}\, e_2;$$

To verify that this point lies in both lines, we can take its exterior product with each of the lines and show the result to be zero.

$\mathcal{G}[\{L_1 \wedge P, L_2 \wedge P\}]$

$\{0, 0\}$

In the special case in which the lines are parallel, that is b c - a d = 0, their intersection is no longer a point, but a *vector defining their common direction*.

The intersection of a line and a plane in a 3-plane

Suppose we wish to find the point **P** of intersection of a line **L** and a plane **Π** in a 3-plane. We express the line as the exterior product of two points in two coordinate planes, and the plane as the exterior product of the points of intersection of the plane with the coordinate axes.

First declare the basis of the 3-plane, and then define the line and plane.

$\mathbb{P}_3$

$\{\mathbb{O}, e_1, e_2, e_3\}$

$$\mathbf{L} = (\mathbb{O} + a\,e_1 + b\,e_2) \wedge (\mathbb{O} + c\,e_2 + d\,e_3);$$
$$\Pi = (\mathbb{O} + e\,e_1) \wedge (\mathbb{O} + f\,e_2) \wedge (\mathbb{O} + g\,e_3);$$

Next, take the regressive product of the line and the plane and simplify it.

$$\mathcal{G}[\mathbf{L} \vee \Pi]$$

$$\mathbb{O} \wedge e_1 \wedge e_2 \wedge e_3 \vee ((d\,e\,f - (b\,e - c\,e + a\,f)\,g)\,\mathbb{O} + a\,e\,(d\,f + (c - f)\,g)\,e_1 + f\,(b\,e\,(d - g) + c\,(-a + e)\,g)\,e_2 - d\,(b\,e + (a - e)\,f)\,g\,e_3)$$

This shows that the intersection point $\mathbf{P}$ is given by:

$$\mathbf{P} = \mathbb{O} + \frac{a\,e\,(d\,f + (c - f)\,g)}{d\,e\,f - (b\,e - c\,e + a\,f)\,g}\,e_1 + \frac{f\,(b\,e\,(d - g) + c\,(-a + e)\,g)}{d\,e\,f - (b\,e - c\,e + a\,f)\,g}\,e_2 - \frac{d\,(b\,e + (a - e)\,f)\,g}{d\,e\,f - (b\,e - c\,e + a\,f)\,g}\,e_3;$$

To verify that this point lies in both the line and the plane, we can take its exterior product with each of the lines and show the result to be zero.

$$\mathcal{G}[\{\mathbf{L} \wedge \mathbf{P},\ \Pi \wedge \mathbf{P}\}]$$

$$\{0,\ 0\}$$

In the special case in which the line is parallel to the plane, their intersection is no longer a point, but a vector defining their common direction. When the line lies in the plane, the result from the calculation will be zero.

The intersection of two planes in a 3-plane

Suppose we wish to find the line $\mathbf{L}$ of intersection of two planes Π_1 and Π_2 in a 3-plane.

First declare the basis of the 3-plane, and then define the planes.

$$\mathbb{P}_3$$

$$\{\mathbb{O},\ e_1,\ e_2,\ e_3\}$$

$$\Pi_1 = (\mathbb{O} + a\,e_1) \wedge (\mathbb{O} + b\,e_2) \wedge (\mathbb{O} + c\,e_3);$$
$$\Pi_2 = (\mathbb{O} + e\,e_1) \wedge (\mathbb{O} + f\,e_2) \wedge (\mathbb{O} + g\,e_3);$$

Next, take the regressive product of the two lines and simplify it.

$$\mathcal{G}[\Pi_1 \vee \Pi_2]$$

$$\mathbb{O} \wedge e_1 \wedge e_2 \wedge e_3 \vee (\mathbb{O} \wedge (a\,e\,(c\,f - b\,g)\,e_1 + b\,f\,(-c\,e + a\,g)\,e_2 + c\,(b\,e - a\,f)\,g\,e_3) + a\,b\,e\,f\,(-c + g)\,e_1 \wedge e_2 + a\,c\,e\,(b - f)\,g\,e_1 \wedge e_3 + b\,c\,(-a + e)\,f\,g\,e_2 \wedge e_3)$$

This shows that the line of intersection $\mathbf{L}$ is given by:

$$\mathbf{L} = \mathbb{O} \wedge (a\,e\,(c\,f - b\,g)\,e_1 + b\,f\,(a\,g - c\,e)\,e_2 + c\,g\,(b\,e - a\,f)\,e_3)$$
$$+ a\,b\,e\,f\,(g - c)\,e_1 \wedge e_2$$
$$+ a\,c\,e\,g\,(b - f)\,e_1 \wedge e_3$$
$$+ b\,c\,f\,g\,(e - a)\,e_2 \wedge e_3;$$

To verify that this line lies in both planes, we can take its exterior product with each of the planes and show the result to be zero.

$$\mathcal{G}[\{\Pi_1 \wedge \mathbf{L},\ \Pi_2 \wedge \mathbf{L}\}]$$

$$\{0,\ 0\}$$

In the special case in which the planes are parallel, their intersection is no longer a line, but a *bivector defining their common 2-direction*.

Example: The osculating plane to a curve

◆ The problem

Show that the osculating planes at any three points to the curve defined by:

$$\mathbf{P} == \mathbb{O} + u\,e_1 + u^2\,e_2 + u^3\,e_3$$

intersect at a point coplanar with these three points.

◆ The solution

The osculating plane Π to the curve at the point $\mathbf{P}$ is given by $\Pi == \mathbf{P} \wedge \mathbf{P}' \wedge \mathbf{P}''$, where $\mathbf{u}$ is a scalar parametrizing the curve, and $\mathbf{P}'$ and $\mathbf{P}''$ are the first and second derivatives of $\mathbf{P}$ with respect to $\mathbf{u}$.

$$\Pi = \mathbf{P} \wedge \mathbf{P}' \wedge \mathbf{P}'';$$

$$\mathbf{P} = \mathbb{O} + u\,e_1 + u^2\,e_2 + u^3\,e_3;$$

$$\mathbf{P}' = e_1 + 2\,u\,e_2 + 3\,u^2\,e_3;$$

$$\mathbf{P}'' = 2\,e_2 + 6\,u\,e_3;$$

We can declare the space to be a 3-plane and the parameter $\mathbf{u}$ (and subscripts of it) to be a scalar, and then use `GrassmannSimplify` to derive the expression for the osculating plane as a function of $\mathbf{u}$.

$$\mathbb{P}_3;\ \mathbf{DeclareExtraScalars}[\{u,\ u\_\}];\ \mathcal{G}[\Pi]$$

$$2\,\mathbb{O} \wedge e_1 \wedge e_2 + 6\,u\,\mathbb{O} \wedge e_1 \wedge e_3 + 6\,u^2\,\mathbb{O} \wedge e_2 \wedge e_3 + 2\,u^3\,e_1 \wedge e_2 \wedge e_3$$

Now select any three points on the curve $\mathbf{P_1}$, $\mathbf{P_2}$, and $\mathbf{P_3}$.

$$P_1 = \mathbb{O} + u_1\, e_1 + u_1^2\, e_2 + u_1^3\, e_3;$$
$$P_2 = \mathbb{O} + u_2\, e_1 + u_2^2\, e_2 + u_2^3\, e_3;$$
$$P_3 = \mathbb{O} + u_3\, e_1 + u_3^2\, e_2 + u_3^3\, e_3;$$

The osculating planes at these three points are:

$$\Pi_1 = \mathbb{O} \wedge e_1 \wedge e_2 + 3\, u_1\, \mathbb{O} \wedge e_1 \wedge e_3 + 3\, u_1^2\, \mathbb{O} \wedge e_2 \wedge e_3 + u_1^3\, e_1 \wedge e_2 \wedge e_3;$$
$$\Pi_2 = \mathbb{O} \wedge e_1 \wedge e_2 + 3\, u_2\, \mathbb{O} \wedge e_1 \wedge e_3 + 3\, u_2^2\, \mathbb{O} \wedge e_2 \wedge e_3 + u_2^3\, e_1 \wedge e_2 \wedge e_3;$$
$$\Pi_3 = \mathbb{O} \wedge e_1 \wedge e_2 + 3\, u_3\, \mathbb{O} \wedge e_1 \wedge e_3 + 3\, u_3^2\, \mathbb{O} \wedge e_2 \wedge e_3 + u_3^3\, e_1 \wedge e_2 \wedge e_3;$$

The point of intersection of these three planes may be obtained by calculating their regressive product.

$$\mathcal{G}[\Pi_1 \vee \Pi_2 \vee \Pi_3]$$

$$\mathbb{O} \wedge e_1 \wedge e_2 \wedge e_3 \vee \mathbb{O} \wedge e_1 \wedge e_2 \wedge e_3 \vee (-9\, \mathbb{O}\, (u_1 - u_2)\, (u_1 - u_3)\, (u_2 - u_3) + 3\, e_1\, (u_1 - u_2)\, (u_1 - u_3)\, (-u_1 - u_2 - u_3)\, (u_2 - u_3) - 9\, e_3\, u_1\, (u_1 - u_2)\, u_2\, (u_1 - u_3)\, (u_2 - u_3)\, u_3 + 3\, e_2\, (u_1 - u_2)\, (u_1 - u_3)\, (u_2 - u_3)\, (-u_2\, u_3 - u_1\, (u_2 + u_3)))$$

This expression is congruent to the point of intersection which we write more simply as:

$$Q = \mathbb{O} + \frac{1}{3}\, (u_1 + u_2 + u_3)\, e_1 + \frac{1}{3}\, (u_1\, u_2 + u_2\, u_3 + u_3\, u_1)\, e_2 + (u_1\, u_2\, u_3)\, e_3;$$

Finally, to show that this point of intersection Q is coplanar with the points P_1, P_2, and P_3, we compute their exterior product.

$$\mathcal{G}[P_1 \wedge P_2 \wedge P_3 \wedge Q]$$

$$0$$

This proves the original assertion.